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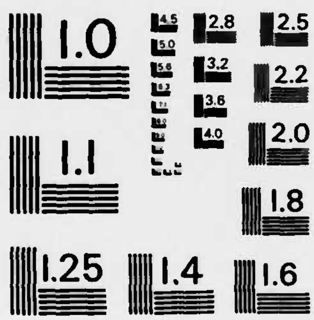

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ABSTRACT

Let $\underline{X} = (\underline{X}_1,\ldots,X_n)$ be independent random variables with logarithmically concave symmetric densities. We show that for any logarithmically concave functions f and g on R^n that are invariant under sign changes,

$$\mathrm{Cov}(f(\underline{X}),g(\underline{X})) \geq 0.$$

Bounds on the values of logarithmically concave densities on R^n evaluated at the mean vector are also given.

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I. INTRODUCTION AND SUMMARY.

The area of positive and negative dependence of multivariate distributions has attracted the attention of many authors over the past decade. (See Abdel-Hameed and Sampson (1978), Jogdeo (1977), Kanter (1975), Karlin and Rinott (1980), Dykstra (1980), and Prekopa (1973).) Pitt (1977) proves that if n(x) is the standard normal density on R^2 and if A and B are balanced convex subsets of R^2 (i.e., $A=-A$ and $B=-B$) then

$$\int_{A\cap B} n(x)dx \geq \left(\int_A n(x)dx\right)\left(\int_B n(x)dx\right).$$

The question whether Pitt's result is true for any standard normal density on R^n, $n > 2$, remains unanswered.

In this paper we investigate covariance inequalities for a class of logarithmically concave densities. We show that if $X_1, \ldots, X_n$ are independent random variables each having a symmetric logarithmically concave symmetric density, then the random variables $Y_1 = f(X_1,\ldots,X_n)$ and $Y_2 = g(X_1,\ldots,X_n)$ are positively correlated whenever f and g belong to a certain class of logarithmically concave functions on R^n. In particular it follows that if A and B are subsets of R^n that are symmetric along all the axes, then

$$\int_{A\cap B} h(x)dx \geq \left(\int_A h(x)dx\right)\left(\int_B h(x)dx\right),$$

where h is the joint density of the independent random variables $X_1, \ldots, X_n$. We remark that a subset of R^n that is symmetric along all the axes is necessarily a balanced set and it follows that if

$(X_1,\ldots,X_n)$ is a standard normal vector, with density n then

$$\int_{A\cap B} n(x)dx \ge \left(\int_A n(x)dx\right)\left(\int_B n(x)dx\right)$$

for all subsets A and B of R^n that are symmetric along all axes.

Throughout, the word "symmetric" will be used to mean "symmetric about the origin". For n=1, 2, ..., let

$H_n = \{f: R^n \to R_+$, f is logarithmically concave and symmetric$\}$,

$A_n = \{A$: A is a $n\times n$ diagonal matrix with diagonal elements $\pm 1\}$,

$G_n = \{f\in H_n;\ f(\underline{x}A) = f(\underline{x})$ for all $\underline{x}$ in R_n and all $A\in A_n\}$, and

$L_n = \{K$: K is a convex symmetric subset of $R^n\}$.

<u>Section 2</u>. Positive Correlations of Functions of Multivariate Random Variables with Logarithmically Concave Densities.

In this section we will show that if X_1, ..., X_n are independent random variables each having a logarithmically concave symmetric density, then the random variables $Y_1 = f(X_1,\ldots,X_n)$ and $Y_2 = g(X_1,\ldots,X_n)$ are positively correlated whenever f and g belong to G_n.

<u>2.1 Theorem</u>. Let H be a convex subset of R^n. Then $f: H\to R_+$ is in H_n if and only if the set $H^+ = \{(\underline{x},z): f(\underline{x}) \ge e^z\}$ is a convex subset of R^{n+1} and $\{\underline{x} : f(\underline{x}) \ge a\}$ is a symmetric subset of R^n for each $a\in R_+$.

<u>Proof</u>. (If) Suppose that f is not in H_n. Then f is either not logarithmically concave or not symmetric. First assume that f is not logarithmically concave on H. Then there exists $\underline{x}_1$, $\underline{x}_2$ in H and a in (0,1) such that

$$f(a\underline{x}_1 + (1-a)\underline{x}_2) < f^a(\underline{x}_1)\, f^{1-a}(\underline{x}_2).$$

Thus the point $(a\underline{x}_1+(1-a)\underline{x}_2,\ a\,\ell n\, f(\underline{x}_1)+(1-a)\ell n\, f(\underline{x}_2)$ belongs to the line segment joining $(\underline{x}_1, \ell n\, f(\underline{x}_1), (\underline{x}_2, \ell n\, f(\underline{x}_2))$ but not in H^+. Since $(\underline{x}_1, \ell n\, f(\underline{x}_1))$ and $(\underline{x}_2, \ell n\, f(\underline{x}_2))$ are in H^+, then H^+ is not convex.

If f is not symmetric on H, then there exists $\underline{x}_0$ in H such that $f(\underline{x}_0) \neq f(-\underline{x}_0)$. Let $a_0=(f(\underline{x}_0) \vee f(-\underline{x}_0))$ and $K_{a_0}=\{\underline{x}:f(\underline{x}) \geq a_0\}$. Then either $\underline{x}_0$ or $-\underline{x}_0$ is in K_{a_0} but not both, contradicting the assumption that K_{a_0} is symmetric.

(Only if). Let $f \in H_n$, $(\underline{x}_1, z_1)$, $(\underline{x}_2, z_2)$ be any two points in H^+ and ℓ is the line joining them. Let $(\underline{x}, z_2)$ be any two points in H^+ and ℓ is the line joining them. Let $(\underline{x}, z)$ be any point on ℓ. Then there exists $0 \leq a \leq 1$ such that

$$\underline{x} = a\underline{x}_1 + (1-a)\underline{x}_2,$$
$$\underline{z} = az_1 + (1-a)z_2.$$

Since $f(\underline{x}_1) \geq e^{z_1}$ and $f(x_2) \geq e^{z_2}$, it follows that $f(\underline{x}) \geq e^{z}$. Therefore, $(\underline{x},z)$ is in H^+. Thus H^+ is convex.

The fact that $\{\underline{x}:f(\underline{x}) > a\}$ is symmetric for each $a \in R_+$ is immediate since f is symmetric.||

<u>2.2 Corollary</u>. Let H be a convex subset of R^n and assume that $f:H \to R_+$ is in H_n. Then, $\{\underline{x}:f(\underline{x}) \geq a\}$ is a convex and symmetric subset of R^n for each $a \in R_+$.

<u>Proof</u>: The symmetry of the set $\{\underline{x}:f(\underline{x}) \geq a\}$ is proved in Theorem 2.1 . Now assume that $\underline{x}_1$, $\underline{x}_2$ are in $\{\underline{x}:f(\underline{x}) \geq a\}$. Then for a in (0,1) we have

$$f(a\underline{x}_1 + (1-a)\underline{x}_2) \geq f^a(\underline{x}_1)\, f^{1-a}(\underline{x}_2) \geq a.$$

Thus, $a\underline{x}_1+(1-a)\underline{x}_2$ is in the set and hence it is convex; since a is arbitrary, the result follows.||

The converse to Corollary 2.2 is not true:

Let f be a concave symmetric function on R which is not in H_1. Then the sets $\{\underline{x}: f(\underline{x}) \geq a\}$ are convex symmetric subsets of R^1. However f is not in H_1.

The following lemma is due to Hoeffding and is a restatement of Lemma 2 of Lehmann [1966].

2.3 Lemma. Let X and Y be extended-valued random variables. Then

$$\text{Cov}(X,Y) = \int_{R^2} \text{Cov}\{I_{X^{-1}[x,\infty]}, I_{Y^{-1}[y,\infty]}\}dxdy,$$ where the

equality is valid even if one side is infinite.

2.4 Lemma. For any $K \in L_{n-1}$, $h \in H_{n-1}$, and $f \in H_n$, the function $g: R \to R_+$ defined by

$$g(x) = \int_K f(x_1,\ldots,x_{n-1},x)\, h(x_1,\ldots,x_{n-1})\, dx_1\ldots dx_{n-1}$$

is in H_1.

Proof: The logconcavity of g follows from Theorem 6 of Prékopa [1973]. The symmetry of g follows from the symmetry of f, h, and k. Thus g is in H_1, as desired.||

2.5 Theorem. Let f and g be in H_1. Suppose that (Ω,F,P) is a probability space and X is an extended-value random variable defined on (Ω,F,P). Then for each f, g in H_1 we have

$$\text{Cov}(f(X),g(X)) \geq 0.$$

Proof: By Lemma 2.3 we have

$$\text{Cov}(f(X),g(X)) = \int_{R^2} \text{Cov}\{I_{[x,\infty]}\circ f(X), I_{[Y,\infty]}\circ g(X)\}dxdy$$

since

$$I_{[x,\infty]}\circ f(X) = I_{f^{-1}[x,\infty]}\circ X.$$

From Corollary 2.2 we know that there exists a constant $a > 0$ such that $f^{-1}[x,\infty] = [-a,a]$. Thus there exists an $a > 0$ such that

$$I_{[x,\infty]} \circ f(X) = I_{[-a,a]} \circ X$$

Similarly we conclude that there is a $b > 0$ such that

$$I_{[y,\infty]} \circ g(X) = I_{[-b,b]} \circ X.$$

Therefore,

$$\mathrm{Cov}(I_{[x,\infty]} \circ f(X), I_{[y,\infty]} \circ g(X)) \text{ equals}$$

$$P\{X \in \{-[a \wedge b],[a \wedge b]\}\} - P\{X \in [-a,a]\} \cdot P\{X \in [-b,b]\},$$

which is clearly nonnegative. From Lemma 2.3 it follows that $\mathrm{Cov}(f(X),\ g(X))$ must be nonnegative.||

__2.6 Theorem__. Let $X_1, \ldots, X_n$ be independent random variables each having a density that belongs to H_1. Then for all f and g in G_n we have $\mathrm{Cov}(f(\underline{X}),\ g(X)) \geq 0$.

__Proof__: We proceed by induction on n. By Theorem 2.5 the result is true for $n=1$. Now assume the result is true for some n_0. For f and g in H_{n_0+1} and $\underline{X} = (X_1,\ldots,X_{n_0+1})$, we write

$$\mathrm{Cov}(f(\underline{X},g(\underline{X})) = E[\mathrm{Cov}\{f(X),g(\underline{X})|X_{n_0+1}\}]$$
$$+ \mathrm{Cov}[Ef(\underline{X}|X_{n_0+1}),\ Eg(\underline{X}|X_{n_0+1})].$$

For a fixed x_{n_0+1}, the function $f_{x_{n_0+1}} : R^n \to R_+$ is defined for any $\underline{x} = (x_1,\ldots,x_{n_0})$ by $f_{x_{n_0+1}}(x_1,\ldots,x_{n_0}) = f(x_1,\ldots,x_{n_0+1})$. For $\underline{x}_1 = (x_{11},\ldots,x_{1n_0})$, $\underline{x}_2 = (x_{21},\ldots,x_{2n_0})$, we let $\underline{x}_1^* = (x_{11},\ldots,x_{1n_0}, x_{n_0+1})$ and $\underline{x}_2^* = (x_{21},\ldots,x_{2n_0},x_{n_0+1})$. Then for a in $(0,1)$,

$$f_{x_{n_0+1}}(a\underline{x}_1+(1-a)\underline{x}_2) = f(a\underline{x}_1^*+(1-a)\underline{x}_2^*) \geq f^a(\underline{x}_1^*)f^{1-a}(\underline{x}_2^*)$$
$$= f^a_{x_{n_0+1}}(\underline{x}_1)\ f^{1-a}_{x_{n_0+1}}(\underline{x}_2)$$

Moreover, for any matrix A in A_{n_0} and $\underline{x} = (x_1,\ldots,x_{n_0})$ in R^{n_0} and $\underline{x}^* = (x_1,\ldots,x_{n_0},x_{n_0+1})$, we have $f_{x_{n_0+1}}(\underline{x}A) = f(\underline{x}^*A^*)$, where A^* = the $(n_0+1) \times (n_0+1)$ diagonal matrix defined by $A^* = \binom{A0}{01}$.

Since f is in G_{n_0+1}, then we have that

$$\begin{aligned} f_{x_{n_0+1}}(\underline{x}A) &= f(\underline{x}^*A^*) \\ &= f(\underline{x}^*) \\ &= f_{x_{n_0+1}}(\underline{x}). \end{aligned}$$

Therefore $f_{x_{n_0+1}}(x_1,\ldots,x_{n_0})$ is in G_{n_0}. The induction hypothesis combined with the above argument gives

$$\begin{aligned} &E[\mathrm{Cov}\{f(\underline{X}),g(\underline{X})\} \mid X_{n_0+1} = x_{n_0+1}] = \\ &E[\mathrm{Cov}\{f_{x_{n_0+1}}(\underline{X}^*),\ g_{x_{n_0+1}}(\underline{X}^*)\}] \geq 0 \end{aligned}$$

where $\underline{x}^* = (x_1,\ldots,x_{n_0})$ when $\underline{x} = (x_1,\ldots,x_{n_0+1})$.

From Lemma 2.4 and the hypothesis, we deduce that $Ef(\underline{X}|X_{n_0+1} = x_{n_0+1})$ as well as $Eg(\underline{X}|X_{n_0+1} = x_{n_0+1})$ are in H_1. By Theorem 2.5 we have $\mathrm{Cov}\{Ef(\underline{X}|X_{n_0+1}),\ Eg(\underline{X}|X_{n_0+1})\} \geq 0$. Thus we finally conclude that $\mathrm{Cov}(f(\underline{X}),g(\underline{X})) \geq 0$. ||

The proof of the following theorem can be obtained by imitating the proof of Theorem 2.6 .

2.7 Theorem. Let $X = (X_1,\ldots,X_n)$ be a standard normal vector with density n. Then

$$\int_{A\cap B} n(x)dx \geq \left(\int_A n(x)dx\right)\left(\int_B n(x)dx\right).$$

for all subsets A, B that are symmetric along all the axes.

Section 3. Bounds on Logarithmically Concave Densities.

In this section we derive some inequalities for strongly unimodal densities. First we define for $n=1,2,\ldots,$

$U_n = \{f:R^n \to R_+;$ f is a logarithmically concave density$\}$.

3.1 Lemma. Let f and g be functions mapping R^n into R_+. Then

$$\int f \ln(f/g) \geq (\int f) \ln(\int f/\int g).$$

In particular, if f is a density function on R^n, then

$$\int f \ln(f/g) \geq -\ln \int g,$$

for any measurable function $g:R^n \to R_+$.

Proof: First assume $\int f = \int g$. Then

$$\int f \ln(f/g) = -\int f \ln(g/f) \geq -\int f\{(g/f)-1\}$$

$$[\text{since } \ln x \leq (x-1),\ x \geq 0]$$

$$= \int f - \int g = 0.$$

Thus the inequality is satisfied in this case.

Now, if $\int f \neq \int g$, then define $g^* = (\int f/\int g)g$, and note that $\int f = \int g^*$. Therefore, using the above inequality we have $\int f \ln(f/g^*) \geq 0$. Using the definition of g^* and simplifying we get $\int f \ln(f/g) \geq (\int f) \ln(\int f/fg)$.||

3.2 Theorem. Let $\underline{X}$ be a random vector with density f belonging to U_n with finite mean $\underline{\mu}$. Let $g:R^n \to R$ be such that $\int \exp(-g) = 1$, then $f(\underline{\mu}) \geq \exp(-\int gf)$.

Proof: Take $f_1=f$, $f_2=e^{-g}$. By Lemma 3.1, we have $\int f_1 \ln(\frac{f_1}{f_2}) \geq 0$. Take $f_1=f$, $f_2=e^{-g}$. Using Jensen's inequality and the fact that f belongs to U_n, we deduce that

$$0 \le \int f\{\ell n\ f + g\} \le \ell n\ f(\mu) + \int fg,$$

completing the proof. ||

<u>3.3 Theorem</u>. Let $\underline{X}$ be a nonnegative random vector with density f belonging to U_n with finite mean $\underline{\mu}$. Then $f(\underline{\mu}) \ge \frac{1}{\mu_1 x \ldots x \mu_n} e^{-n}$. Equality is attained for $f(\underline{x}) = \prod_{i=1}^{n} \frac{1}{\mu_i} e^{-x_i/\mu_i}$ and therefore the bound is sharp.

<u>Proof</u>: Choose $g(x) = \sum_i \log \frac{\mu_i}{a_i} + \sum_i a_i \frac{x_i}{\mu_i}$ for $\underline{x} \ge 0$. Then $\int e^{-g} = \prod_{i=1}^{n} \int_0^\infty \frac{a_i}{\mu_i} e^{-a_i x_i/\mu_i} dx_i = 1$. Also $\int gf = \Sigma \log \frac{\mu_i}{a_i} + \Sigma a_i$. Thus by Theorem 3.2, $f(\underline{\mu}) \ge \prod_{i=1}^{n} (\frac{a_i}{\mu_i} e^{-a_i})$. The right hand side is maximized by choosing $a_i = 1$, $i = 1, \ldots, n$. Equality is attained for

$f(\underline{x}) = \prod_{i=1}^{n} \frac{1}{\mu_i} e^{-x_i/\mu_i}$, as may be directly verified. ||

The following theorem gives a lower bound on the peak of density function belonging to U_n in terms of the determinant of its covariance matrix.

<u>3.4 Theorem</u>. Let $\underline{X}$ be a multivariate vector with mean $\underline{\mu}$, covariance matrix Σ, and density f belonging to U_n. Then

$$f(\mu) \ge (2\pi)^{-n/2} |\Sigma|^{-1/2} \exp(-n/2).$$

<u>Proof</u>: Let $g(\underline{x}) = \frac{1}{2} \ell n |\Sigma| + (n/2) \ell n (2\pi) + \frac{1}{2}(\underline{x}-\underline{\mu})' \Sigma^{-1} (\underline{x}-\underline{\mu})$.

Then $\int e^{-g} = 1$, and $\int g(\underline{x}) f(\underline{x}) d\underline{x} = Eg(X) = \frac{1}{2}\ell n |\Sigma| + (n/2) \ell n (2\pi) + n/2$. The desired conclusion then follows from Theorem 3.2 . ||

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